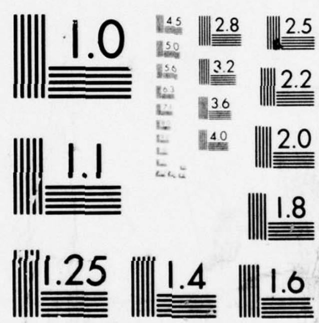
AD-A060 417 FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS F/G 12/1
CHARACTERIZATION OF NONPARAMETRIC CLASSES OF LIFE DISTRIBUTIONS--ETC(U)
JUL 78 N A LANGBERG, R V LEON, F PROSCHAN AFOSR-76-3109
UNCLASSIFIED FSU-STATISTICS-M473 AFOSR-TR-78-1395 NL

1 OF 1

AD
A060417

END
DATE
FILMED
1-79
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AFOSR-TR- 78-1395

(18) (19)

(2) **LEVEL** II

AD A060417

DDC FILE COPY

(6) CHARACTERIZATION OF NONPARAMETRIC CLASSES OF LIFE DISTRIBUTIONS,

(9) Interim rept.,

by

(10) Naftali A. Langberg[1], Ramón V. León[2,3] [redacted] Frank Proschan[3]

(15) AFOSR-76-3109, AFOSR-74-2581

FSU Statistics Report M473
AFOSR Technical Report No. 86
AFOSRN Technical Report No. 11

(11) Jul[redacted]978 (12) 23 p.

The Florida State University
Department of Statistics
Tallahassee, Florida 32306

(14) FSU-STATISTICS-M473, TR-86

[1]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR 76-3109.

[2]Research sponsored by the National Institute of Environmental Health Sciences, under Grant 5 T32 ES07011.

[3]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant 74-2581.

78 10 16 160

400 277

Characterization of Nonparametric Classes of Life Distributions

by

Naftali A. Langberg[1], Ramón V. León[2,3] and Frank Proschan[3]

ABSTRACT

In this paper we obtain characterizations of large classes of nonparametric life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, etc., classes. The methods used differ from the usual functional equation methods used for the far more common characterizations of parametric families of life distributions.

[1]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant AFOSR 76-3109.

[2]Research sponsored by the National Institute of Environmental Health Sciences, under Grant 5 T32 ES07011.

[3]Research sponsored by the Air Force Office of Scientific Research, AFSC, USAF, under Grant 74-2581.

Characterization of Nonparametric Classes of Life Distributions

by

Naftali A. Langberg, Ramón V. León, and Frank Proschan

1. Introduction and Summary. Characterizations of particular parametric families of life distributions are quite common in the literature (see, for example, Kagen, Linnik, and Rao, 1973, and Patil, Kotz, and Ord, 1975). In this paper, by contrast, we present characterizations of large classes of nonparametric life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, etc., classes. (See Section 2 for exact definitions.) Such characterizations are far less common and generally require quite different mathematical and statistical techniques.

Our characterizations are based on order statistics, weighted spacings between order statistics, and total time on test transforms; in most cases inequalities among limiting expected values determine the characterizations. Related results concerning total time on test transforms had been obtained earlier by Barlow and colleagues (exact references are given for each of these results as they appear in the text below), but not necessarily under the weakest assumptions on the distributions. Since in characterization, emphasis is placed on obtaining results under the weakest assumptions on the distributions being characterized, we have found it useful to prove stronger versions of a number of these known results--for example, a characterization which requires that a distribution be differentiable is not as appealing as one that requires that it only be continuous.

In Section 2, we present preliminaries consisting of definitions and notation. In Section 3, we present properties of the total time on test transform and a characterization of the IFR(DFR) class of life distributions in terms of the concavity (convexity) of the total time on test transform. In Section 4, we present characterizations of the IFR(DFR) classes based on the monotonicity of the expected values of the weighted spacings between successive order statistics; the number of sample sizes required is infinite. By using the fact that the exponential distribution is both IFR and DFR, we are able to obtain a strengthened characterization of the exponential distribution, as compared with the earlier Saleh (1976) characterization. We also obtain additional characterizations of the IFR(DFR) distribution requiring only a single sample size; of course, we must compensate by making the stronger assumption of stochastic monotonicity rather than expected value monotonicity. In Section 5, we present characterizations of distributions such as IFRA, NBU, NBUE, and their duals, which are similar in spirit to those in Section 4 for the IFR(DFR) classes.

One final remark should be made. Chandra and Singpurwalla (1978) have pointed out the close raltionship between the total time on test transform and the Lorenz curve used by econometrists. Thus, some of our results of Section 3 concerning total time on test transforms can be used to obtain analogous results for the Lorenz curve, and may thus be of interest and value in fields other than reliability.

2. Preliminaries. Let F be a life distribution, that is $F(0-) = 0$. We use the following notation and conventions: $F^{-1}(t) \equiv \inf\{x: F(x) > t\}$, $t \in [0, 1)$; $F^{-1}(1) \equiv \sup\{x: F(x) < 1\}$; $\overline{F} \equiv 1 - F$; $R \equiv -\ln \overline{F}$. We use "increasing" in place of "nondecreasing" and "decreasing" in place of "nondecreasing".

Next we define the classes of life distributions to be considered in the sequel.

Definition 2.1. (a) F is increasing failure rate (IFR) if $\overline{F}(t + x)/\overline{F}(t)$ is decreasing in t $(-\infty < t < F^{-1}(1))$ for each $x > 0$.

(b) F is (shifted) decreasing failure rate (SDFR) if $\overline{F}(t + x)/\overline{F}(t)$ is increasing in $t(F^{-1}(0) \le t < \infty)$ for each $x > 0$.

(c) F is increasing failure rate average (IFRA) if $\frac{1}{t}R(t)$ is increasing in t $(0 < t < F^{-1}(1))$.

(d) F is decreasing failure rate average (DFRA) if $\frac{1}{t}R(t)$ is decreasing in $t > 0$.

(e) F is new better than used (NBU) if $\overline{F}(x + y) \le \overline{F}(x)\,\overline{F}(y)$ for $x > 0$, $y > 0$.

(f) F is new worse than used (NWU) if $\overline{F}(x + y) \ge \overline{F}(x)\,\overline{F}(y)$ for $x > 0$, $y > 0$.

(g) F is new better than used in expectation (NBUE) if (i) $\int_0^\infty x\,dF(x) < \infty$; (ii) $\int_t^\infty \overline{F}(x)dx \ge (\int_0^\infty x\,dF(x))\overline{F}(t)$ for $t > 0$.

(h) F is new worse than used in expectation (NWUE) if $\int_t^\infty \overline{F}(x)dx \le (\int_0^\infty x\,dF(x))\overline{F}(t)$ for $t > 0$.

The chain of implications IFR $\Rightarrow$ IFRA $\Rightarrow$ NBU $\Rightarrow$ NBUE is readily established (see Marshall and Proschan, 1972).

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from F. The k-th <u>weighted spacing</u>, $W_{k:n}$, between order statistics $X_{k-1:n}$ and $X_{k:n}$ is defined by $W_{k:n} \equiv (n - k + 1) \cdot (X_{k:n} - X_{k-1:n})$ for k = 1, 2, ..., n, where $X_{0:n} \equiv 0$. The <u>total time on test</u> up to the k-th order statistic, $T(X_{k:n})$, is defined by $T(X_{k:n}) = \sum_{i=1}^{k} W_{i:n}$ for k = 1, 2, ..., n, and $T(X_{0:n}) \equiv 0$. If we assume that n items are placed on test at time 0 and that successive failures are observed at times $X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$, then $W_{k:n}$ represents the total test time observed between $X_{k-1:n}$ and $X_{k:n}$, and $T(X_{k:n})$ represents the total test time observed between 0 and $X_{k:n}$ (see Barlow and Proschan, 1975, p. 61).

3. Properties of the Total Time on Test and Its Transform. Let $H_F^{-1}(t) \equiv \int_0^{F^{-1}(t)} \overline{F}(u)du$ for $0 \le t \le 1$. Barlow and Campo (1975) call H_F^{-1} the total time on test transform. In this section we develop some of the properties of H_F^{-1}.

Before starting the first theorem we need two definitions.

Definition 3.1. A point x is a point of increase of F if $F(x - h) < F(x) < F(x + h)$ for every $h > 0$.

Definition 3.2. A sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$ of ordered pairs of natural numbers is a t-sequence $(0 \le t \le 1)$ if (i) $1 \le k_r \le n_r < n_r + 1$ for all r, and (ii) $k_r/n_r \to t$ as $r \to \infty$.

Theorem 3.3. Let $F^{-1}(t)$ be a point of increase of F, and let (k, n) range over a t-sequence. Then as $n \to \infty$,

$$\frac{1}{n} T(X_{k:n}) \to H_F^{-1}(t) \text{ a.s. .}$$

Proof. Let F_n denote the empirical distribution function of F. Then $T(X_{k:n}) = n\, H_{F_n}^{-1}\left(\frac{k-1}{n}\right) = \int_0^{X_{k:n}} \overline{F}_n(u)du$ (see Barlow and Campo, 1975). Also for (k, n) ranging over a t-sequence, $X_{k:n} \to F^{-1}(t)$ a.s. as $n \to \infty$ since $F^{-1}(t)$ is a point of increase of F (see Rao, 1973, p. 423). The desired result follows by the Glivenko-Cantelli Theorem (Chung, 1974, p. 123). ||

Next we note that if EX_1 is finite, then $EX_{k:n}$, $EW_{k:n}$, and $ET(X_{k:n})$ are also finite, since $0 \le X_{k:n} \le T(X_{k:n}) \le \sum_{i=1}^{n} X_i \equiv n\,\overline{X}_n$. This observation can be used to show that whenever EX_1 is finite, $\{\frac{1}{n_r} T(X_{kr:nr})\}_{r=1}^{\infty}$ is uniformly integrable for every t-sequence $\{(k_r, n_r)\}_{r=1}^{\infty}$. Since a uniformly convergent sequence which converges almost surely converges in mean (see Breiman, 1973, p. 91), we can state the following result.

Theorem 3.4. Let t, k, and n be as in Theorem 3.3 and let EX_1 be finite. Then $E|\frac{1}{n}T(X_{k:n}) - H_F^{-1}(t)| \to 0$ as $n \to \infty$. In particular, $\frac{1}{n}ET(X_{k:n}) \to H_F^{-1}(t)$ as $n \to \infty$.

We remark that neither Theorem 3.3 nor Theorem 3.4 is true if t is not a point of increase of F. In this case a counterexample to Theorem 3.3 and to Theorem 3.4 can easily be constructed using the facts that $\varliminf_{n\to\infty} X_{[nt]:n} = F^{-1}(t-)$ and $\varlimsup_{n\to\infty} X_{[nt]:n} = F^{-1}(t-)$ and that $F^{-1}(t-) \neq F^{-1}(t)$. ($[\cdot]$ denotes the greatest integer function). To show that $\varliminf_{n\to\infty} X_{[nt]:n} = F^{-1}(t)$ and $\varlimsup_{n\to\infty} X_{[nt]:n} = F^{-1}(t)$, use the fact that $P[X_{[nt]:n} > x] = P[B(n, \overline{F}(x)) > n - [nt] + 1]$, where $B(n, \overline{F}(x))$ denotes a binomial random variable (see Mood, Graybill, and Boes, 1974, p. 252), and the law of the iterated logarithm (see Breiman, 1968, p. 291).

Let ${}^{+}f(x_0)$ denote the right-hand derivative of f at the point x_0. We have the following lemma.

Lemma 3.5. Let x be a point of increase and of continuity of F. Then ${}^{+}H_F^{-1}(F(x))$ exists and is nonzero if and only if ${}^{+}R(x)$ exists and is nonzero. In either case, ${}^{+}H_F^{-1}(F(x))\ {}^{+}R(x) = 1$.

Proof. Note that in a neighborhood of x, F^{-1} behaves like the usual inverse function of F. The result follows using standard differentiation results. ||

The next proposition is easily verified.

Proposition 3.6. The life distribution F is (i) IFR if and only if either F is degenerate or R(x) is convex on $(F^{-1}(0), F^{-1}(1))$ and $F(F^{-1}(0)) = 0$; (ii) SDFR if and only if R(x) is concave on $(F^{-1}(0), \infty)$.

The following simple properties of H_F^{-1} will be needed in the proof of our next theorem.

(3.1) $H_F^{-1}(0) = F^{-1}(0)$.

(3.2) $H_F^{-1}(t +) = H_F^{-1}(1) = EX_1$.

(3.3) $H_F^{-1}(1 -) = H_F^{-1}(1) = EX_1$.

(3.4) H_F^{-1} is increasing on [0, 1].

(3.5) For $y \in [0, \infty)$, the set $\{s: H_F^{-1}(s) = y\} = [a, b)$, where $0 \leq a < b \leq 1$, if and only if $P(X = F^{-1}(a)) = b - a$.

(3.6) For $0 \leq a < 1$, $H_F^{-1}(a -) = H_F^{-1}(a)$ if and only if $F(F^{-1}(a -)) = F(F^{-1}(a))$. In particular H_F^{-1} is continuous on [a, b) if and only if every point in $(F^{-1}(a), F^{-1}(b -))$ is a point of increase of F.

Theorem 3.7. (Barlow and Campo, 1975). The life distribution F is IFR(SDFR) if and only if H_F^{-1} is concave (convex) on [0, 1].

Proof. Let H_F^{-1} be concave on [0, 1]. Since H_F^{-1} is increasing on [0, 1], there exists a real number A in [0, 1] such that H_F^{-1} is strictly increasing on [0, A] and constant on [A, 1]. If A = 0, H_F^{-1} is constant on [0, 1] and consequently, F is the IFR distribution degenerate at $F^{-1}(0)$. Next suppose that A > 0. It follows that $H_F^{-1}(t) > 0$ for all $t \in (0, A)$. Equivalently, ${}^{+}H(F(x)) > 0$ for x in $(F^{-1}(0), F^{-1}(1))$ since $F^{-1}(A-) = F^{-1}(1)$. By (3.5) and (3.6), every point of $(F^{-1}(0), F^{-1}(1))$ is a point of increase and of continuity of F. Hence, by Lemma 3.5, the concavity of H_F^{-1} implies that ${}^{+}R(x)$ exists and is increasing on $(F^{-1}(0), F^{-1}(1))$, that is, R is convex on $(F^{-1}(0), F^{-1}(1))$. Since by (3.6) $F(F^{-1}(0)) = 0$, then F is IFR by Proposition 3.6.

Next let F be IFR. By Proposition 3.6 either F is degenerate in which case H_F^{-1} is constant and thus concave or R(x) is convex on $S \equiv (F^{-1}(0), F^{-1}(1))$ and $R(F^{-1}(0)) = 0$. Assuming the latter, let $x \in S$ and $h > 0$. Then

$$0 < \frac{R(x) - R(F^{-1}(0))}{x - F^{-1}(0)} \le \frac{R(x + h) - R(x)}{h} .$$

Consequently ^{+}R exists and is positive on S. Now since $^{+}F = \bar{F}\ ^{+}R$ is positive on S, S contains only points on increase and of continuity of F. Thus by Lemma 3.5, $^{+}H_F^{-1} \circ F$ is decreasing on S, that is, $^{+}H_F^{-1}$ is decreasing on (0, 1). Hence H_F^{-1} is concave on (0, 1). Since an IFR distribution has a finite mean by (3.2) and (3.3), H_F^{-1} is concave on [0, 1].

The counterpart results for the SDFR case can be proved similarly. ||

Theorem 3.7 is due to Barlow and Campo, 1975 (see also Barlow, 1977), but our proof is new. Our proof avoids some technical difficulties which arise in the limiting argument used in the Barlow and Campo proof of the "if" part of Theorem 3.7.

4. Characterizations of the IFR(SDFR) Class. Barlow and Proschan (1966) have shown that if F is IFR(SDFR), then for all $x > 0$, $P(W_{k:n} > u)$ is decreasing (increasing) in k ($k = 2, 3, \ldots, n$) for all $n \geq 2$. If F has a finite mean, then $E\, W_{k:n} < \infty$ for all choices of k and n and consequently $E\, W_{k:n}$ is decreasing (increasing) in $k(k = 2, 3, \ldots, n)$ for all $n \geq 2$. In this section we prove that a slightly weaker version of the last condition is sufficient for F to be IFR(SDFR). Then we use this result to obtain a characterization of the shifted exponential obtained by Saleh (see Kotz, 1974), who required stronger regularity conditions on F than we require. Two other characterizations of the IFR(SDFR) are given.

The main result of this section follows.

Theorem 4.1. Let F be a continuous life distribution with finite mean. Then F is IFR(SDFR) if and only if $E\, W_{k:n}$ is decreasing (increasing) in $k(k = 2, \ldots, n)$ for infinitely many n.

The "only if" part has already been shown. To prove the "if" part, we need the following lemma. This lemma shows that every point in the support of F is a point of increase of F. Thus Theorem 3.4 can be used at every point of the support of F to show that H_F^{-1} is concave (convex), that is, F is IFR(SDFR) by Theorem 3.7. We remark that this proof avoids all assumptions on F other than continuity. [If we are willing to assume the existence of a positive continuous density f everywhere in the support of F, we can obtain a more direct proof of the "if" part of Theorem 4.1 by using the fact that in this case $W_{[nt]:n}$ converges in distribution to an exponential distribution with failure rate $r(F^{-1}(t))$, where $r(t) \equiv \dfrac{f(t)}{\overline{F}(t)}$ is the failure rate of F (see Pyke, 1965).]

<u>Lemma 4.2</u>. Let F be a continuous life distribution with finite mean. Let $E\, W_{k:n}$ be decreasing (increasing) in k (k = 2, ..., n) for infinitely many n. Then the support of F is the interval $[F^{-1}(0), F^{-1}(1)]$.

<u>Proof</u>. The support of a continuous distribution is a closed set without isolated points (see Chung, 1974, p. 10). It follows that if S, the support of F, is not an interval we can find a, b, and ε such that $(a - \varepsilon, a] \subset S$, $(a, b) \subset \sim S \equiv \{x: x \notin S\}$, and $[b, b + \varepsilon) \subset S$. Let $t = F(a) = F(b)$, $t_1 = \frac{t + F(a - \varepsilon)}{2}$ and $t_2 = (t + F(b + \varepsilon))/2$. Also let $h > 0$ be small enough so that $[t_1 - h, t_1 + h] \subset (F(a - \varepsilon), t)$ and $[t_2 - h, t_2 + h] \subset (t, F(b + \varepsilon))$. Since $T(X_{k:n}) \equiv \sum_{i=1}^{k} W_{i:n}$, we obtain for each one of the infinitely many n that

$$
\begin{aligned}
& ET(X_{([n(t_1-h)]+[n\cdot 2h]):n}) - ET(X_{[n(t_1-h)]:n}) \\
(4.1) \qquad & \geq(\leq)\ ET(X_{([n(t-h)]+[n\cdot 2h]):n}) - ET(X_{[n(t-h)]:n}) \\
& \geq(\leq)\ ET(X_{([n(t_2-h)]+[n\cdot 2h]):n}) - ET(X_{[n(t_2-h)]:n}).
\end{aligned}
$$

The points at which F equals $t_1 - h$, $t_1 + h$, $t - h$, $t + h$, $t_2 - h$, and $t_2 + h$ are all in the interior of S and are consequently points of increase of F. Applying Theorem 3.4 to the chain of inequalities (4.1), we conclude that

$$
\begin{aligned}
& H_F^{-1}(t_1 + h) - H_F^{-1}(t_1 - h) \\
(4.2) \qquad & \geq(\leq)\ H_F^{-1}(t + h) - H_F^{-1}(t - h) \\
& \geq(\leq)\ H_F^{-1}(t_2 + h) - H_F^{-1}(t_2 - h).
\end{aligned}
$$

Since $H_F^{-1}(\cdot) = \int_0^{F^{-1}(\cdot)} \overline{F}(u)du$ is continuous at t_1 and t_2, letting $h \to 0$ in (4.2), we conclude that $\lim_{h\downarrow 0} H_F^{-1}(t + h) - H_F^{-1}(t - h) = 0$. But since H_F^{-1} is increasing, this implies that H_F^{-1} is continuous at t, or equivalently, that F^{-1} is continuous at t. This contradicts the fact that F is constant on (a, b). It follows that S must be an interval. The desired result follows. ||

We now complete the proof of Theorem 4.1.

Proof of Sufficiency. Let t_1, t_2, and h be such that $0 \le t_1 < t_2 < t_2 + h \le 1$. Using the argument in the proof of Lemma 3.8 yielding (4.2), we obtain:

$$H_F^{-1}(t_1 + h) - H_F^{-1}(t_1) \ge (\le)\ H_F^{-1}(t_2 + h) - H_F^{-1}(t_2). \tag{4.3}$$

Since (4.3) is true for all t_1, t_2, and h satisfying the constraints above, H_F^{-1} must be concave (convex) on [0, 1]. By Theorem 3.7, this implies that F is IFR(DFR). ||

It is clear from the proof of Theorem 4.1 that the following characterization of the IFR(SDFR) class is also true.

Theorem 4.3. Let F be a continuous life distribution with finite mean. Then F is IFR(SDFR) if and only if for infinitely many $n \ge N$ and some $\ell(1 \le \ell < N)$

$$E \sum_{i=k}^{k+\ell} W_{i:n}$$

is decreasing (increasing) in $k(1 \le k \le n - \ell)$.

Note that F is both IFR and SDFR if and only if F is shifted exponential, that is,

$$\overline{F}(x) = \begin{cases} e - \lambda(x - F^{-1}(0)) & x \ge F^{-1}(0) \\ 1 & x < F^{-1}(0), \end{cases}$$

for some positive λ. Hence as a corollary of Theorem 4.1 we obtain the following:

Corollary 4.4. Let F be a continuous life distribution with finite mean. Then F is shifted exponential with mean μ if and only if for infinitely many $n \ge 2$, $E\, W_{k:n} = \mu$ for $k = 2, 3, \ldots, n$.

A similar characterization was obtained by Saleh (1976) but with the additional (unnecessary) condition that $\inf\{x\colon\ F(x) \ge t\}$ is differentiable on $(0, 1)$.

Now we give another characterization of the IFR(SDFR) class which requires conditions for only one sample size. Since

$$(4.4) \qquad P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x) = (\overline{F}(x + u)/\overline{F}(x))^{n-m}$$

(see David, 1970, p. 18), the following result holds:

Theorem 4.5. The distribution F is IFR(SDFR) if and only if for some fixed n and m $(2 \le m + 1 \le n)$, and all $u \ge 0$, $P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x)$ is decreasing (increasing) in $x(-\infty < x < F^{-1}(1))$ $[(F^{-1}(0) \le x < \infty)]$.

Actually if F is IFR(SDFR), then $P(X_{m+1:n} - X_{m:n} > u \mid X_{m:n} = x)$ is decreasing (increasing) in x for all n and m, where $2 \le m + 1 \le n$. However since the emphasis of this paper is on characterizations, we omit this generalization from the statement of Theorem 4.5. A similar remark can be made about other theorems in this paper (see Theorem 5.1 for example).

Recall that a random variable X is stochastically increasing (decreasing) in Y, another random variable, if for all x, $P(X > x \mid Y = y)$ is increasing (decreasing) in y. Hence Theorem 4.5 can be restated using this language. Similar remarks apply to other theorems in this paper (see for example Theorems 5.1 and 5.3).

5. Characterizations of Classes of Life Distributions Other Than the IFR(SDFR). In this section we characterize classes of life distributions other than the IFR(SDFR). The following two characterizations are similar in spirit to the characterization of the IFR(SDFR) class given in Theorem 4.5. By (4.4) we immediately obtain:

Theorem 5.1. The life distribution F is NBU(NWU) if and only if $P(X_{1:n-m} > u) \geq(\leq) P(X_{m+1:n} - X_{m:n} > u | X_{m:n} = x)$ for some fixed n and m ($1 \leq m < n$), and all $u \geq 0$ and $x \geq 0$.

Since $E(X_{n:n} - X_{n-1:n} | X_{n-1:n} = x) = (\int_x^\infty \bar{F}(u)du)/\bar{F}(x)$, we also have:

Theorem 5.2. Let F be a life distribution with finite mean. Then F is NBUE (NWUE) if and only if $E(X_{n:n} - X_{n-1:n} | X_{n-1:n} = x) \leq(\geq) EX_1$ for some fixed $n \geq 2$, and all $0 \leq x < F^{-1}(1)$.

Next we give a characterization of the IFRA(DFRA) class:

Theorem 5.3. Let F be a life distribution such that $F(0) = 0$. Then F is IFRA if and only if for all $x > 0$, $P(W_{1:n} > x)$ is increasing in $n \geq N$, where N is arbitrary.

Proof. We have for $0 \leq x < \infty$, $P(W_{1:n} > x) = \bar{F}^n(x/n)$.

It follows that $P(W_{1:n} > x)$ is increasing in $n \geq N$ for all $x > 0$ if and only if

$$(5.1) \qquad \bar{F}^n(x/n) \leq \bar{F}^m(x/m) \text{ for all } N \leq n < m \text{ and all } 0 \leq x < \infty.$$

Recall that F is IFRA if and only if

$$(5.2) \qquad \bar{F}^{1/t_2}(t_2) \leq \bar{F}^{1/t_1}(t_1) \text{ for all } 0 < t_1 < t_2 < F^{-1}(1).$$

We show that (5.1) implies (5.2). Let $0 < t_1 < t_2$, both be rational, and let $N \leq n < m$ be natural numbers such that $\frac{t_2}{t_1} = \frac{m}{n}$. Let $\alpha = (m_1 t_1)^{-1}$.

Then $n\alpha = 1/t_2$ and $m\alpha = 1/t_1$, so it is easily seen that (5.1) implies that $\overline{F}^{1/t_2}(x\alpha t_2) \le \overline{F}^{1/t_1}(x\alpha t_1)$. Letting $x = \alpha^{-1}$, (5.2) follows since $\overline{F}$ is right continuous and the rationals are dense.

To show (5.2) implies (5.1), let x, n, and m be such that $x \ge 0$ and $N \le n < m$. If $x = 0$, then $\overline{F}^n(x/n) = \overline{F}^m(x/m) = 1$ since $F(0) = 0$ so (5.1) follows. If $x/m \ge F^{-1}(1)$ then $\overline{F}(x/m) = 1$ and (5.1) also follows. If $0 < x$ and $\frac{x}{m} < F^{-1}(1)$, then set $t_1 = x/m$ and $t_2 = x/n$ in (5.2) to obtain (5.1). ||

The following dual theorem has a similar proof which we omit.

<u>Theorem 5.4.</u> Let F be a life distribution. Then F is DFRA if and only if for all $x > 0$, $P(W_{1:n} > x)$ is decreasing in $n \ge N$, where N is arbitrary.

To prove our next result we need a theorem of Barlow and Proschan (1966). Let X(Y) have distribution F(G). We assume that $F(0) = 0 = G(0)$, and that F and G are continuous. We also assume that the support of F is an interval, possibly infinite, and that G is strictly increasing on its support.

<u>Theorem 5.5.</u> (Barlow and Proschan, 1966, Theorem 3.6). Let $\frac{G_0^{-1}F(x)}{x}$ be increasing in x in the support of F. Then $EX_{i:n}/EY_{i:n}$ is decreasing in i ($i = 1, 2, \ldots, n$).

<u>Theorem 5.6.</u> Let F be a continuous life distribution with finite mean. Assume that the support of F is an interval and that $F(0) = 0$. Then F is IFRA(DFRA) if and only if $EX_{i:n}/(\sum_{k=1}^{i} 1/(n - k + 1))$ is decreasing (increasing) in i ($i = 1, 2, \ldots, n$) for infinitely many n.

<u>Proof.</u> Let G in Theorem 5.5 be the exponential distribution with mean 1. Then $EY_{i:n} = \sum_{k=1}^{i} \frac{1}{n - k + 1}$ (see Barlow and Proschan, 1975, p. 60). Thus necessity follows from Theorem 5.5 (as does the dual result for the opposite direction of monotonicity) if we note that $G^{-1}(x) = -\ell n(1 - x)$.

To prove sufficiency, first observe that every point in the support of $F(G)$ is a point of increase of $F(G)$. Hence $Y_{[nt]:n} \to G^{-1}(t) = -\ln(1 - t)$ and $X_{[nt]:n} \to F^{-1}(t)$ a.s. as $n \to \infty$. We show $\{X_{[nt]:n}\}_{n=1}^{\infty}$ is uniformly integrable. We have

$$P(X_{[nt]:n} > x) = P(B(n, \overline{F}(x)) > n - [nt] + 1),$$

where $B(n, \overline{F}(x))$ denotes a binomial random variable with parameters n and $\overline{F}(x)$. Thus

$$P[X_{[nt]:n} \geq \frac{n}{n - [nt] + 1} \overline{F}(x) \tag{5.3}$$

since $P(Z > t) \leq \frac{EZ}{t}$ for any nonnegative random variable Z. Hence

$$\begin{aligned} &EX_{[nt]:n}\ I[X_{[nt]:n} \geq A] \\ &= \int_A^{\infty} P[X_{[nt]:n} > x]dx + AP[X_{[nt]:n} \geq A] \\ &\quad [\text{by integration by parts}] \\ &\leq \frac{n}{n - [nt] + 1}\left(\int_A^{\infty} \overline{F}(x)dx + A\,\overline{F}(A)\right) \\ &\quad [\text{by (5.3)}] \\ &\leq \frac{1}{1 - \frac{[nt]}{n} + \frac{1}{n}}\left(E\,X_1\ I[X_1 \geq A]\right). \end{aligned}$$

It follows that $X_{[nt]:n}$ (and similarly $Y_{[nt]:n}$) is a uniformly integrable sequence in n. Consequently, $EX_{[nt]:n} \to F^{-1}(t)$ and $EY_{[nt]:n} \to G^{-1}(t)$ as $n \to \infty$. Thus by hypothesis, $F^{-1}(t)/(-\ln(1 - t))$ is decreasing (increasing) in t $(0 < t < 1)$. Equivalently, $\frac{F^{-1}(F(x))}{-\ln(1 - \overline{F}(x))} = \frac{x}{-\ln\overline{F}(x)}$ is decreasing (increasing) in x $(0 < x < F^{-1}(1))$. Sufficiency follows. ||

Now we characterize the NBU (NWU) class. First we prove a lemma.

Lemma 5.7. Let F be a continuous life distribution. Then F is NBU (NWU) if and only if $F^{-1}(t + s - ts) \le (\ge) F^{-1}(t) + F^{-1}(s)$ for every t and s such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F.

Proof. Let $F^{-1}(t + s - ts) \le F^{-1}(t) + F^{-1}(s)$ for all t and s such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F. To show that F is NBU, it is enough to show that $\overline{F}(x + y) \le \overline{F}(x)\, \overline{F}(y)$ for all x and $y \in D = \{z > 0: F(z - \varepsilon) < F(z)$ for all $\varepsilon > 0\}$. But since $\overline{F}$ is continuous for each x and $y \in D$, we can find sequences $\{x_n\}$ and $\{y_n\}$ of points of increase of F such that $x_n \uparrow x$ and $y_n \uparrow y$ as $n \to \infty$. Thus it is enough to show $\overline{F}(x + y) \le \overline{F}(x)\, \overline{F}(y)$ for all x and y which are points of increase of F. Let x and y be two such points and let $t = F(x)$ and $s = F(y)$. Then $t + s - ts \le F(F^{-1}(t + s - ts)) \le F(F^{-1}(t) + F^{-1}(s))$ by hypothesis. Hence $1 - t - s + ts \ge \overline{F}(F^{-1}(t) + F^{-1}(s))$; that is, $\overline{F}(x)\, \overline{F}(y) \ge \overline{F}(x + y)$, as desired.

Necessity follows by reversing the steps of the last three sentences above. ||

Theorem 5.8. Let F be a continuous life distribution with finite mean. Then F is NBU (NWU) if and only if for every t and s in $(0, 1)$ such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F, we have

$$E(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} | X_{[nt]:n}) \overset{\text{a.s.}}{\le (\ge)} EX_{[n(1-t)s]:n-[nt]}$$

for infinitely many n.

Proof. We first prove sufficiency. Let t and s be such that $F^{-1}(t)$ and $F^{-1}(s)$ are points of increase of F, let n always range over the infinite sequence of the hypothesis. By the Markov property of order statistics, we have

$$(5.4)\qquad P_n(x) \equiv P(X_{[nt]+[n(1-t)s]:n} - X_{[nt]} > x \mid X_{[nt]:n})$$

$$\overset{a.s.}{=} P\left(B\left(n - [nt], \frac{\overline{F}(x + X_{[nt]:n})}{\overline{F}(X_{[nt]:n})}\right) > n - [nt] - [n(1 - t)s] + 1\right),$$

where B(n, p) is a binomial random variable with parameters n and p.

But since $F^{-1}(t)$ is a point of increase of F, then by the SLLN and the fact that $X_{[nt]:n} \to F^{-1}(t)$ a.s. as $n \to \infty$, we have

$$(5.5)\qquad \frac{1}{n - [nt]} B\left(n - [nt], \frac{\overline{F}(x + X_{[nt]:n})}{\overline{F}(X_{[nt]:n})}\right) \to \frac{\overline{F}(x + F^{-1}(t))}{1 - t}$$

a.s. as $n \to \infty$. Hence by (5.4) and (5.5),

$$P(X_{[nt]+[n(1 - t)s]:n} - X_{[nt]:n} > x \mid X_{[nt]:n})$$

$$(5.6)\qquad \to \begin{cases} 1 & \text{if } x < F^{-1}(t + s - ts) - F^{-1}(t) \\ 0 & \text{if } x > F^{-1}(t + s - ts) - F^{-1}(t) \end{cases}$$

a.s. as $n \to \infty$.

Now (5.4) and the inequality $P(Z > t) \le \frac{EZ}{t}$ $(t > 0)$ imply that

$$P_n(x) \overset{a.s.}{\le} \frac{1}{n - [nt] - [n(1 - t)s] + 1}\left(\frac{(n - [nt])\overline{F}(x + X_{[nt]:n})}{\overline{F}(X_{[nt]:n})}\right)$$

$$\overset{a.s.}{\le} \left(\frac{(n - [nt])}{(n - [nt] - [n(1 + t)s]\,\overline{F}(X_{[nt]:n})}\right)\overline{F}(x).$$

Since $\int_0^\infty \overline{F}(x)dx < \infty$, $X_{[nt]} \to F^{-1}(t)$ a.s. as $n \to \infty$, and $\overline{F}(F^{-1}(t)) > 0$, it follows that for all sufficiently large n, $P_n(x) \overset{a.s.}{\le} g(x)$, where g(x) is integrable. Thus by the Lebesgue dominated convergence theorem,

$$E(X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n}) = \int_0^\infty P_n(x)dx$$

$$\to F^{-1}(t + s - ts) - F^{-1}(t) \text{ a.s. as } n \to \infty.$$

Since $X_{[n(1-t)s]:n-[nt]} \to F^{-1}(s)$ a.s. as $n \to \infty$, we have by hypothesis that $F^{-1}(t + s - ts) \le F^{-1}(t) + F^{-1}(s)$. Sufficiency follows from Lemma 5.7.

To show necessity, let X(H) denote a random variable with distribution H. For $y \ge 0$, let $\overline{G}_y \equiv \overline{F}(x + y)/\overline{F}(y)$. Then F is NBU (NWU) if and only if for all $y > 0$, $X(G_y)$ is stochastically smaller (larger) than X(F), written $X(G_y) \overset{st}{\le} (\overset{st}{\ge}) X(F)$. Hence if F is NBU (NWU), then for all $y \ge 0$, $0 < t < 1$, and $0 < s < 1$,

$$X(G_y)_{[n(1-t)s]:n-[nt]} \overset{st}{\le} (\overset{st}{\ge}) X(F)_{[n(1-t)s]:n-[nt]}.$$

But by the Markov property of order statistics, the conditional random variable $X_{[nt]+[n(1-t)s]:n} - X_{[nt]:n} | X_{[nt]} = y$ has the same distribution as the random variable $X(G_y)_{[n(1-t)s]:n-[nt]}$ necessity follows. ||

Observe that by the Markov property of order statistics,

$$\frac{1}{n - [nt]} E\left(\sum_{k=1}^{n-[nt]} X_{[nt]+k:n} - X_{[nt]:n} | X_{[nt]:n}\right)$$

$$\overset{a.s.}{=} \frac{\int_0^\infty \overline{F}(x + X_{[nt]:n})ds}{\overline{F}(X_{[nt]:n})}.$$

Hence we can use the methods of the proof of Theorem 5.8 to obtain the following characterization of the NBUE (NWUE) class.

Theorem 5.9. Let F be a continuous life distribution with finite mean. Then F is NBUE (NWUE) if and only if for every t in (0, 1) such that $F^{-1}(t)$ is a point of increase, we have

$$\frac{1}{n - [nt]} E\left(\sum_{k=1}^{n-[nt]} X_{[nt]+k:n} - EX_{[nt]:n} \middle| X_{[nt]:n} \right) \le EX_1$$

for infinitely many n.

REFERENCES

1. Barlow, R. E. (1977). Geometry of the total time on test transform. Technical Report ORC 77-11, University of California, Berkeley.

2. Barlow, R. E. and Campo, R. (1975). Total time on test processes and applications to failure data analysis. Reliability and Fault Tree Analysis, 451-482 (R. E. Barlow, J. B. Fussell and N. D. Singpurwalla, eds.). SIAM, Philadelphia.

3. Barlow, R. E. and Proschan, F. (1966). Inequalities for linear combinations of order statistics from restricted families. Ann. Math. Statist. 37, 1574-1591.

4. Barlow, R. E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing: Probability Models. Holt, Rinehart and Winston, Inc., New York.

5. Breiman, Leo (1968). Probability. Addison-Wesley Publishing Company, Reading, Massachusetts).

6. Chandra, Mahesh, and Singpurwalla, N. D. (1978). On the Gini index, the Lorenz curve, and the total time on test transform. George Washington University Serial T-368.

7. Chung, K. L. (1974). A Course in Probability Theory, 2nd ed.. Academic Press, New York.

8. David, H. A. (1970). Order Statistics. John Wiley and Sons, New York.

9. Goldie, C. M. (1977). Convergence theorems for empirical Lorenz curves and their inverses. Adv. Appl. Prob. 9, 765-791.

10. Kagan, A. M., Linnik, Yu. V., Rao, C. R. (1972). Characterization Problems in Mathematical Statistics. Nawka (in Russian), Moscow. (English translation by B. Ramachandran. Wiley, 1973).

11. Kotz, S. (1974). Characterizations of statistical distributions: a supplement to recent surveys. Rev. Inst. Internat. Statist. 42, 39-65.

12. Mood, A. M., Graybill, F. A., and Boes, D. C. (1963). Introduction to the Theory of Statistics, 3rd. ed. McGraw-Hill, New York.

13. Patil, G. P., Kotz, S. and Ord, J. K., eds. (1975). Statistical Distribution in Scientific Work, Volume 3 - Characterizations and Applications. D. Reidel, Dordrecht.

14. Pyke, R. (1965). Spacings. J. Roy. Statist. Soc., Ser. B, 7, 395-449.

15. Rao, C. R. (1973). Linear Statistical Inference and Its Applications. John Wiley and Sons, New York.

16. Saleh, A. K. Md. E. (1976). Characterization of distributions using expected spacings between consecutive order statistics. J. of Statist. Res. 10, 1-13.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER AFOSR-TR- 78-1395	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CHARACTERIZATION OF NONPARAMETRIC CLASSES OF LIFE DISTRIBUTIONS		5. TYPE OF REPORT & PERIOD COVERED Interim
		6. PERFORMING ORG. REPORT NUMBER FSU No. M473
7. AUTHOR(s) Naftali A. Langberg, Ramon Leon and Frank Proschan		8. CONTRACT OR GRANT NUMBER(s) AFOSR 76-3109
9. PERFORMING ORGANIZATION NAME AND ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61102F 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS Air Force Office of Scientific Research/NM Bolling AFB, Washington, DC 20332		12. REPORT DATE July 1978
		13. NUMBER OF PAGES 21
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION ST. ENT (of abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY TES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Characterizations, life distributions, order statistics, spacings, total time on test transform, nonparametric, Lorenz curve, failure rate, reliability.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In this paper we obtain characterizations of large classes of nonparametric life distributions, such as the increasing (decreasing) failure rate, increasing (decreasing) failure rate average, new better (worse) than used, etc., classes. The methods used differ from the usual functional equation methods used for the far more common characterizations of parametric families of life distributions.

DD FORM 1 JAN 73 1473

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)